# Boiling an Egg

Consider a spherical homogeneous egg of specific heat capacity $c$, density $\rho$, thermal conductivity $\kappa$ and radius $a$ (all constants). Its initial temperature is $T_{\text{egg}}$, and it is cooked by immersion in water at time $t = 0$ which keeps the surface temperature constant at $T_{\text{water}}$ thereafter.

## The Thermal Diffusion Equation

A spherical shell of material, radius $r < a$ and thickness $\delta r$, has a radial thermal resistance $\delta R$ and total heat capacity $\delta C$ given by

$$\delta C = \rho 4\pi r^2 c\delta r \quad \text{and} \quad \delta R = \delta r / 4\pi r^2 \kappa. \tag{1,2}$$

Conservation of energy requires that, in terms of these quantities,

$$T(r + \delta r, t) - T(r,t) = \dot{Q}(r,t)\delta R(r,t) \tag{3}$$

$$\left[T(r,t+\delta t) - T(r,t)\right]\delta C = \left[\dot{Q}(r-\delta r,t) - \dot{Q}(r,t)\right]\delta t \tag{4}$$

and taking limits as $\delta r \to 0$ and $\delta t \to 0$ gives

$$\frac{\mathrm{d}r}{\mathrm{d}R}\left(\frac{\partial T}{\partial r}\right) = 4\pi\kappa r^2\left(\frac{\partial T}{\partial r}\right) = \dot{Q}(r,t) \quad \text{and} \quad \frac{\mathrm{d}C}{\mathrm{d}r}\left(\frac{\partial T}{\partial t}\right) = 4\pi c\rho r^2\left(\frac{\partial T}{\partial t}\right) = \left(\frac{\partial \dot{Q}}{\partial r}\right) \tag{5,6}$$

and therefore, eliminating the radial heat flux $\dot{Q}$,

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right) = \frac{\tau_0 r^2}{a^2}\left(\frac{\partial T}{\partial t}\right) \quad \text{where} \quad \tau_0 = c\rho a^2/\kappa \tag{7}$$

which is a special case of the thermal diffusion equation.

## Solving the Diffusion Equation

The solution to equation 7 is found with an expansion in spherical waves of wave number $k$

$$T_k(r,t) = \frac{B_k}{r}\exp\left(i[\omega t - kr]\right) \tag{8}$$

which must satisfy equation 7, *i.e.*

$$-r^2k^2T_k = \frac{\tau_0 r^2}{a^2}\left(i\omega T_k\right) \quad \Rightarrow \quad T_k(r,t) = \frac{B_k}{r}\exp(-ikr)\exp\left(-k^2a^2t/\tau_0\right). \tag{9,10}$$

The physically significant solutions are real and must also be finite at $r = 0$ so

$$T(r,t) = T_{\text{water}} + \left(T_{\text{egg}} - T_{\text{water}}\right)\frac{2a}{\pi r}\sum_{N=1}^{\infty}\frac{(-1)^{N-1}}{N}\sin\left(\frac{N\pi r}{a}\right)\exp\left(\frac{-N^2\pi^2 t}{\tau_0}\right) \tag{11}$$

where the amplitudes have been selected to satisfy the boundary conditions:

$$T(r < a,0) = T_{\text{egg}}; \quad T(a,0) = T_{\text{water}}; \quad T(r,\infty) = T_{\text{water}}. \tag{12a-c}$$

## Applying the Solution

Terms in equation 11 with $N > 1$ decay rapidly so when $t > 0.1\tau_0$

$$T(r,t) \approx T_{\text{water}} + \left(T_{\text{egg}} - T_{\text{water}}\right)\frac{2a}{\pi r}\sin\left(\frac{\pi r}{a}\right)\exp\left(\frac{-\pi^2 t}{\tau_0}\right). \tag{13}$$

We consider the egg to be ‘cooked’ when the yolk–white boundary is at temperature $T_{\text{yolk}}$. As the yolk comprises 33% of the egg this means that $T_{\text{yolk}} = T(0.69a, t_{\text{cooked}})$

$$\frac{\pi 0.69}{2\sin(0.69\pi)}\frac{\left(T_{\text{yolk}} - T_{\text{water}}\right)}{\left(T_{\text{egg}} - T_{\text{water}}\right)} = \exp\left(\frac{-\pi^2 t_{\text{cooked}}}{\tau_0}\right) \tag{14}$$

so

$$t_{\text{cooked}} = \lambda M^{2/3}\log_e\left[0.76\times\frac{\left(T_{\text{egg}} - T_{\text{water}}\right)}{\left(T_{\text{yolk}} - T_{\text{water}}\right)}\right] \quad \text{where} \quad \lambda = \frac{c\rho^{1/3}}{\pi^2\kappa(4\pi/3)^{2/3}} \tag{15,16}$$

Approximate values (derived from S. L. Polley, O. P. Snyder and P. Kotnour, Food Technol. **34(11)** (1980) 76-94.) for the thermal properties needed to calculate $\lambda$ are:

| | $c$ | $\kappa$ | $\rho$ | $\lambda$ |
|---|---|---|---|---|
| Yolk | 2.7 J g$^{-1}$ K$^{-1}$ | $3.4\times10^{-3}$ W cm$^{-1}$ K$^{-1}$ | 1.032 g cm$^{-3}$ | 31 (s g$^{-2/3}$) |
| White | 3.7 J g$^{-1}$ K$^{-1}$ | $5.4\times10^{-3}$ W cm$^{-1}$ K$^{-1}$ | 1.038 g cm$^{-3}$ | 27 (s g$^{-2/3}$) |